Translational invariance in the kicked harmonic oscillator

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Analysis of the quantum kicked harmonic oscillator in the condition of resonance ($\omega_0 T=2\pi/q$) is approached in this paper. We extend the results found in Berman, Rubaev, and Zaslavsky [Nonlinearity **4**, 543 (1991)] by relating the translational invariance of the qth power of the quasienergy operator with the spreading of the mean energy in time. These results are then confirmed by numerical simulations.

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I. INTRODUCTION

In the last decade a lot of work has been done on the classical harmonic kicked oscillator [1], and more recently, much attention has been devoted to its quantum version [2,3]. This system has an unexpected feature: the phase space is spanned by a stochastic web for certain values of the ratio between the frequency of the oscillator (ω_0) and the frequency of the kicks (ω_1), so that a slow diffusion is possible for any value (however small) of the perturbation strength. This is a kind of Arnol'd diffusion in a system having $1\frac{1}{2}$ degrees of freedom. At a more detailed level, it is possible to show that the web exists only if the two frequencies are in a rational ratio and it can have both translational and rotational symmetry (crystal) for $\omega_0/\omega_1=1/q$, $q\in q_c\equiv\{1,2,3,4,6\}$. In all the other rational cases it displays so-called quasisymmetry (quasicrystal). We remind the reader that the values of q belonging to q_c represent the way to fill completely the two-dimensional plane with regular polygons having q sides, even if $q=1,2$ are degenerate cases ($q=3$ are triangles; $q=4$ are squares; $q=6$ are hexagons).

As a consequence [1] of the connected web, the classical model has a diffusive behavior both in the crystal and in the quasicrystal case, even for small values of the perturbation strength (for any small value in the crystal case). However, when the frequency of free motion and that of kick have an irrational ratio, different features are present, which can be summarized as follows [1]: (a) a threshold in the perturbation strength $k_{cr}(E)$ exists for each initial value of energy, or (b) a threshold in the initial value of energy $E_{cr}(k)$ exists for each perturbation strength such that for $k>k_{cr}$ or $E>E_{cr}$ the growth of the unperturbed average energy is possible in a diffusivelike way.

Typical pictures of the phase space surfaces of section show invariant tori close to the origin [1]; thus conditions (a) and (b) merely state that we need to increase the separatrix widths or to avoid the disconnected integrable region in order to have unbounded growth of the average energy.

The quantum version of the kicked harmonic oscillator (QKHO) was analyzed in [2] where a suitable and quite involved operator that commutes with the evolution operator itself was found in resonance condition. In this way the quasienergy eigenfunctions have an additional symmetry, which becomes translational invariance for $q\in q_c$. Therefore the possibility of extended states has been suggested, giving an unbounded growth of the average energy in time. In this paper we make this point more precise by deriving the classical resonance condition for the crystal case as the most general property of commutation of the qth power of the quasienergy operator with the translation operators in the phase plane (the Weyl operators).

Our results can be summarized as follows: we found analytically that the q power of the evolution operator for the resonant case ($q\in q_c$) always commutes with a one or two parameter group of commuting translation operators. Numerically these behaviors show a mean energy spreading in time, respectively, in a diffusive or ballistic way. A suggestive stationary phase argument makes this point clearer, giving an intuitive, although not rigorous, interpretation of the phenomenon.

What happens if the system has no translational invariance? One may argue that the traditional localization picture takes place [4–6] and the energy growth will stop after a break time τ_B. This is really what our computations show, for example, for ω_0/ω_1 irrational or $\omega_0/\omega_1=1/q$ with $q\notin q_c$. These results were also found in [3] for $\omega_0/\omega_1=\frac{1}{5}$, together with a transition from localized to delocalized states for $k>k_{cr}$. A check of this transition is outside the reliability of our computer program. In any case, there is no contradiction with our picture: results from [3] only mean that some other mechanism acts for large k, which is not connected with translational properties.

The paper is organized as follows: in Sec. II we remind the reader of a few fundamental facts about the classical model; in Sec. III we prove that the q power of the evolution operator commutes with the translation operators in the phase space if and only if $q\in q_c$; then we study the commutation properties of the translation operators themselves and find the conditions under which we may have a one or two parameter groups of independent

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translation in the phase space. Section IV is devoted to numerical calculations that confirm our previous results.

II. CLASSICAL RESULTS

We review some classical results from [2]. The classical Hamiltonian

$$H=\frac{p^2}{2}+\frac{1}{2}\omega_0{}^2x^2-k\delta_T(t)\cos x \qquad (2.1)$$

describes a harmonic oscillator kicked with a potential $k\cos x$, δ_T being the usual periodic δ function and T the period of the kicks.

From the Hamiltonian (2.1) the map over a period T follows:

$$\begin{aligned}\bar{x}&=\frac{p}{\omega_0}\sin\alpha-\frac{k}{\omega_0}\sin x\sin\alpha+x\cos\alpha\ ,\\ \bar{p}&=p\cos\alpha-x\omega_0\sin\alpha-k\sin x\cos\alpha\ ,\end{aligned} \qquad (2.2)$$

where $\alpha=\omega_0T$. This map can be written by using the variables $u=p/\omega_0$ and $v=-x$, so that the classical parameters are $k_0=k/\omega_0$ and $\alpha=\omega_0T$. Consequently, the classical mean energy at time t is given by

$$E(t)=\langle p^2+\omega_0^2x^2\rangle_t/2=\omega_0^2\langle v^2+u^2\rangle_t/2\ . \qquad (2.3)$$

If a resonance condition is assumed ($\alpha=2\pi/q$, $q\in\mathbb{Z}$), then for any k value the phase plane is filled with a connected stochastic web. Inside the web the motion is chaotic and diffusion is possible even for small k. This web has both translational and rotational symmetry for $q\in q_c=\{1,2,3,4,6\}$.

Under the resonance condition and for small perturbation, the Hamiltonian can be written as [1]

$$H=H_q+V_q\ ,$$

where

$$\begin{aligned}H_q&=-\frac{1}{q}\frac{k}{\omega_0}\sum_{j=1}^{q}\cos(\vec{R}\cdot\vec{e}_j)\ ,\\ V_q&=-\frac{2}{q}\frac{k}{\omega_0}\sum_{j=1}^{q}\cos(\vec{R}\cdot\vec{e}_j)\sum_{m=1}^{\infty}\cos\left[\frac{2\pi m}{q}\left[\frac{t}{T}-j\right]\right]\end{aligned} \qquad (2.4)$$

and

$$\vec{R}\equiv(v,u)\ ,$$

$$\vec{e}_j\equiv\left[\cos\frac{2\pi}{q}j,\sin\frac{2\pi}{q}j\right]\ ,\quad j=1,\ldots,q\ .$$

This form will be useful later.

III. QUANTUM APPROACH

The Floquet operator (the evolution operator over a period T) is given by ($\hbar=1$)

$$\mathcal{U}_T(k)=e^{ik\cos(\hat{x})}e^{-i\hat{H}_0T}\ , \qquad (3.1)$$

where $\hat{H}_0=\hat{p}^2/2+\omega_0^2\hat{x}^2/2$ is the harmonic oscillator Hamiltonian ("free Hamiltonian"). Let us introduce the translation operators in the phase space:

$$\mathcal{T}_{s,r}=e^{i(s\hat{x}+r\hat{p})}\ , \qquad (3.2)$$

expressing the translation of a quantity r along the x axis and s along the p axis (r and s are two real numbers), and the usual creator and annihilation operators, $\hat{a},\hat{a}^\dagger$,

$$\begin{aligned}\hat{a}&=\sqrt{1/2\omega_0}(\omega_0\hat{x}+i\hat{p})\ ,\\ \hat{a}^\dagger&=\sqrt{1/2\omega_0}(\omega_0\hat{x}-i\hat{p})\ ,\end{aligned} \qquad (3.3)$$

with the commutation rule $[\hat{a},\hat{a}^\dagger]=1$. In terms of $\hat{a}$ and $\hat{a}^\dagger$, (3.2) becomes

$$\mathcal{T}_{s,r}=e^{z\hat{a}-z^*\hat{a}^\dagger}\equiv\mathcal{T}(z)\ , \qquad (3.4)$$

where

$$z=\left[r\left[\frac{\omega_0}{2}\right]^{1/2}+\frac{is}{\sqrt{2\omega_0}}\right] \qquad (3.5)$$

and z^* is the usual complex conjugate.

By using standard commutation rules it is easy to show that

$$\begin{aligned}\mathcal{T}_{s,r}\mathcal{U}_T(k)&=\mathcal{T}_{s,r}e^{ik\cos(\hat{x})}e^{-i\hat{H}_0T}\\ &=e^{ik\cos(\hat{x}+r)}\mathcal{T}_{s,r}e^{-i\hat{H}_0T}\\ &=e^{ik\cos(\hat{x}+r)}e^{-i\hat{H}_0T}\mathcal{T}_{s',r'}\ ,\end{aligned} \qquad (3.6)$$

where r' and s' are given by ($\alpha=\omega_0T$),

$$\begin{aligned}r'&=r\cos\alpha+\frac{s}{\omega_0}\sin\alpha\ ,\\ s'&=s\cos\alpha-\omega_0r\sin\alpha\ ,\end{aligned} \qquad (3.7)$$

and

$$\mathcal{T}_{s',r'}=\mathcal{T}(z')=\mathcal{T}(ze^{-i\alpha})\ . \qquad (3.8)$$

By applying q times (3.6) we obtain

$$\begin{aligned}\mathcal{T}_{s_0,r_0}\mathcal{U}_T^q(k)&=e^{ik\cos(\hat{x}+r_0)}e^{-i\hat{H}_0T}\mathcal{T}_{s_1,r_1}\mathcal{U}_T^{q-1}(k)\\ &=\cdots=\prod_{j=0}^{q-1}[e^{ik\cos(\hat{x}+r_j)}\mathcal{U}_T(0)]\mathcal{T}_{s_q,r_q}\ ,\end{aligned} \qquad (3.9)$$

where the parameters s_j,r_j are given by

$$\begin{aligned}r_j&=r_0\cos(j\alpha)+\frac{s_0}{\omega_0}\sin(j\alpha)\ ,\\ s_j&=s_0\cos(j\alpha)-\omega_0r_0\sin(j\alpha)\ ,\quad j=1,\ldots,q\ .\end{aligned} \qquad (3.10)$$

Equation (3.9) can also be written as

$$\begin{aligned}\mathcal{T}(z_0)\prod_{j=0}^{q-1}&[e^{ik\cos(\hat{x})}\mathcal{U}_T(0)]\\ &=\prod_{j=0}^{q-1}[e^{ik\cos(\hat{x}+r_j)}\mathcal{U}_T(0)]\mathcal{T}(z_0e^{-iq\alpha})\ ,\end{aligned} \qquad (3.11)$$

where $z_0 = r_0\sqrt{\omega_0/2} + is_0/\sqrt{2\omega_0}$. Therefore the operators commute if and only if

$$r_j = 2\pi l_j \ , \quad j = 0, \ldots, q-1 \ , \tag{3.12}$$

$$q\alpha = 2\pi l \ , \quad l, l_j \in \mathbb{Z} \ . \tag{3.13}$$

[Note that from (3.13) and (3.10) we obtain $r_q = r_0$ and $s_q = s_0$, as we should.] We would like to remark that if we put $l=1$ in Eq. (3.13) we recover the classical resonance condition.

By coupling Eqs. (3.10), (3.12), and (3.13), we obtain

$$\cos\frac{2\pi j}{q} = \frac{n_j}{m_j} \ , \quad n_j, m_j \in \mathbb{Z} \tag{3.14a}$$

$$s_0 \sin\frac{2\pi j}{q} = 2\pi\omega_0 k_j \ , \quad k_j \in \mathbb{Z} \tag{3.14b}$$

for $j = 1, \ldots, (q-1)/2$ if q is odd, and $j = 1, \ldots, q/2-1$ if q is even. Equations (3.14a) are satisfied if and only if

$$\cos\frac{2\pi}{q} \in \mathbb{Q} \ . \tag{3.15}$$

In fact, if $\cos(2\pi/q)$ is irrational, Eq. (3.14a) for $j=1$ cannot be true. Conversely, suppose that $\cos(2\pi/q) \in \mathbb{Q}$; then we have

$$\begin{aligned} \cos\left[\frac{2\pi j}{q}\right] &= 2^{j-1}\cos^j\left[\frac{2\pi}{q}\right] - j2^{j-3}\cos^{j-2}\left[\frac{2\pi}{q}\right] \\ &\quad + \frac{j}{2}\begin{bmatrix} j-3 \\ 1 \end{bmatrix} 2^{j-5}\cos^{j-4}\left[\frac{2\pi}{q}\right] \\ &\quad + \frac{j}{3}\begin{bmatrix} j-4 \\ 2 \end{bmatrix} 2^{j-7}\cos^{j-6}\left[\frac{2\pi}{q}\right] \\ &\quad + \cdots \ , \end{aligned} \tag{3.16}$$

which is evidently rational.

On the other side, $\cos(2\pi/q) \in \mathbb{Q}$ if and only if [7] $q \in q_c = \{1,2,3,4,6\}$. We can then summarize the previous result as follows: for $q \in q_c$ it is possible to find some r_0 and s_0 such that the qth power of the one period evolution operator does commute with $\mathcal{T}_{s_0,r_0}$. In this case the eigenstates of $\mathcal{U}^q$ are invariant for translations in the phase space so that they have to be extended.

We would like to stress that (3.15) exactly corresponds to the invariance of the classical Hamiltonian (2.4) under translations in both directions in the phase space. Our result show that the classical phase space invariance is reflected in the quantum model by a property of the evolution operator over q periods.

As defined in Eq. (3.2), translation operators in phase space are generally dependent on two parameters. Nevertheless the requirement of independence of different translations forces one to take into account the commuting group of translations. Imposing the property of commutation, we find a one- or two-parameter group depending on ω_0, that is, on $\hbar$.

The cases $q=1$ and $q=2$ are trivial because the evolution operator becomes $e^{ik\cos\hat{x}}$ and $-e^{ik\cos\hat{x}}$, respectively; the spectrum is then absolutely continuous, and the energy $(E = p^2/2 + \omega_0^2 x^2/2)$ grows quadratically in time (ballistic motion).

The $q=4$ case is very interesting since Eqs. (3.12) and (3.14b) read $r_0 = 2l_0\pi$, $s_0 = 2\pi l_1\omega_0$, so that

$$\mathcal{T}_{l_1,l_0} = e^{2\pi i(l_1\omega_0\hat{x} + l_0\hat{p})} \ , \quad l_0, l_1 \in \mathbb{Z} \ . \tag{3.17}$$

For arbitrary $n_1, n_0 \in \mathbb{Z}$, one has

$$\mathcal{T}_{l_1,l_0} \cdot \mathcal{T}_{n_1,n_0} = \mathcal{T}_{n_1,n_0} \cdot \mathcal{T}_{l_1,l_0} e^{4\pi^2 i\omega_0(n_1 l_0 - n_0 l_1)} \ . \tag{3.18}$$

Then they commute only if

$$n_1 l_0 = n_0 l_1 \Longrightarrow l_0/l_1 = n_0/n_1 \equiv s \in \mathbb{Q} \tag{3.19a}$$

or

$$2\pi\omega_0 = n \in \mathbb{Z} \ . \tag{3.19b}$$

Let us mention that since $\omega_0 = 1/\hbar$, this means that $\hbar = 2\pi/n$, namely, a particular kind of rational $\hbar/2\pi$. Condition (3.19a) gives $\forall s \in \mathbb{Q}$ fixed, a one-parameter commuting group

$$\mathcal{T}_n^{(s)} = e^{2\pi i n(s\omega_0\hat{x} + \hat{p})} \quad (n \in \mathbb{Z}) \ , \tag{3.20a}$$

while (3.19b) gives a two-parameter group:

$$\mathcal{T}_{n,l} = e^{i(n\hat{x} + 2\pi l\hat{p})} \ , \quad n, l \in \mathbb{Z} \ . \tag{3.20b}$$

(By a suitable choice of the free variables n, l it is possible to assign a group structure such that $\mathcal{T}_{n_1 l_1} \cdot \mathcal{T}_{n_2 l_2} = \mathcal{T}_{n_1+n_2, l_1+l_2}$.)

Despite the appearance, the case $q=3$ is very close to $q=4$, since from

$$\mathcal{T}_{l_1,l_2} = e^{2\pi i[\omega_0/\sqrt{3}(l_1 - l_2)\hat{x} + (l_1 + l_2)\hat{p}]} \ , \quad l_1, l_2 \in \mathbb{Z} \ ; \tag{3.21}$$

imposing the commutation property, we obtain $\forall s = l_1/l_2 \in \mathbb{Q}$ fixed, $n \in \mathbb{Z}$, the one-parameter group

$$\mathcal{T}_n^{(s)} = e^{2\pi i n(\omega_0/\sqrt{3}(s-1)\hat{x} + (s+1)\hat{p})} \ ; \tag{3.22a}$$

and if $4\pi\omega_0/\sqrt{3} = n \in \mathbb{Z}$, a two-parameter group

$$\mathcal{T}_{l_1,l_2} = e^{i(l_1\hat{x}/2 + 2\pi l_2\hat{p})} \ , \quad l_1, l_2 \in \mathbb{Z} \ . \tag{3.22b}$$

The case $q=6$ is analogous to $q=3$.

In this was we have found that the qth power of the evolution operator in the "crystal" resonant condition $(q \in \{q_c\})$ commutes with a one-parameter group or a two-parameter group of commuting translational operators in the phase plane. In the next section we show both numerically and analytically that these conditions correspond to an algebraic growth of the mean energy in time with an exponent equal to 1 or 2.

Can we extrapolate from the above conclusions any result about the rational $(q \notin q_c)$ or the irrational case? The answer is no. Any search for more involved commutation rules, e.g., $[\mathcal{T}_{n_1,s_1},[\mathcal{T}_{n_2,s_2},\mathcal{U}^q]]=0$, gave a negative answer, and the simple invariance of the qth power of the evolution operator for rotations of angles of $2\pi/q$ does not imply simple relations with the growth of the average energy. However, our numerical computations enforce the usual picture of localization, even if the reliability of the numerical procedure does not allow the investigation of the model for wide ranges of k.

IV. RESULTS

We use the configurational space approach for the construction of the one-period evolution operator:

$$\psi(x,T)=e^{ik\cos x}\int_{-\infty}^{\infty}\mathcal{G}(x,T,x',0)\psi(x',0)dx' , \qquad (4.1)$$

where $\mathcal{G}$ is the Feynman propagator of the harmonic oscillator

$$\mathcal{G}(x,T;x',0)=\left[\frac{\omega_0}{2\pi i\sin(\omega_0 T)}\right]^{1/2}\exp\left\{i\frac{\omega_0}{2\sin(\omega_0 T)}[(x'^2+x^2)\cos(\omega_0 T)-2xx']\right\} . \qquad (4.2)$$

By discretizing a length L in N small intervals Δ, this propagator becomes a $N\times N$ matrix; the request of unitarity imposed $\Delta=\sqrt{2\pi|\sin(\omega_0 T)|/\omega_0 N}$.

We set the parameters in such a way as to realize a translational invariance along one or two directions in the phase space. The relevant quantities under investigation are the mean energy

$$E(t)=\left\langle\psi_t\left|\frac{p^2}{2}+\frac{1}{2}\omega_0^2x^2\right|\psi_t\right\rangle$$

as a function of the discrete time t for the classical and the quantum model, and the contour plot of the Husimi distribution, defined as

$$\mathcal{H}(q,p,t)=|\langle\Phi_{qp}|\psi_t\rangle|^2$$

where Φ_{qp} is the coherent state peaked at the point (q,p) in the classical phase space, and ψ_t is the evoluted wave function at time t.

Numerically meaningful quantum results are obtained by choosing the parameters corresponding to non-fully-chaotic classical phase space. To obtain an effective diffusion, we therefore put the initial classical distribution along the separatrix web or centered around some unstable fixed point. As initial quantum state we take a coherent state peaked inside the classical separatrix.

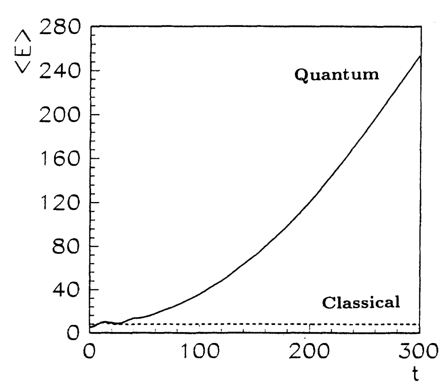

FIG. 1. Average energy versus time for $\omega_0/\omega_1=\frac{1}{4}$, $\omega_0=2/\pi$, and $k=0.5$ (ballistic behavior). Full line, quantum; dashed line, classical.

In the first set of pictures we present the quantum resonant case $\omega_0/\omega_1=\frac{1}{4}$, $\omega_0=2/\pi$. For these parameters the condition of commutation with a two-parameter group (3.19b) is satisfied. Figure 1 shows the quadratic growth of the quantum mean energy compared with the classical one. The contour plot of the Husimi distribution is compared in Fig. 2 with the classical distribution in the phase space (dots). The isotropic spread of the Husimi distribution suggests the following interpretation (as in [8]).

Because of the presence of two Bloch indices we can decompose the time-evoluted wave function in the following way:

$$\psi(x,t)=\sum_{k=1}^{n}\int d\alpha\int d\beta\, e^{i\lambda_k(\alpha,\beta)t}\phi_k(\alpha,\beta,x) , \qquad (4.3)$$

where $\phi_k(\alpha,\beta,x)$ are obtained by decomposing the initial wave function in Bloch waves, and k indicates a possible summation over n bands. By applying the stationary phase method to $\mathcal{H}$ we obtain

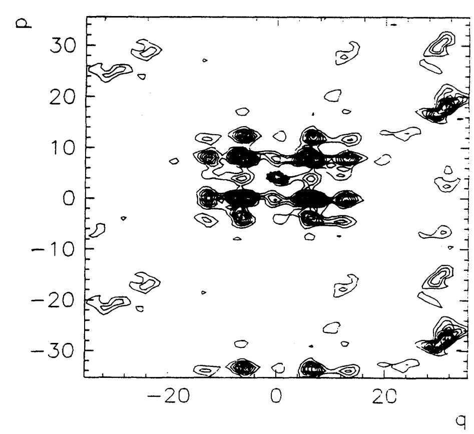

FIG. 2. Contour plot of the Husimi distribution at the time $t=300$ for the same data as in Fig. 1. Dots (in the center of the picture) are the classical distribution function at the same time. As initial quantum state we take a coherent state peaked at the point $(0,\pi)$ in the phase plane, and as initial classical distribution we take $N=200$ points.

$$\mathcal{H}(q,p,t)=\left|\sum_{k=1}^{n}\int dx\,\Phi_{qp}^{*}(x)\int d\alpha\int d\beta e^{i\lambda_k(\alpha,\beta)t}\phi_k(\alpha,\beta,x)\right|^2$$
$$\approx\frac{1}{t^2}\left|\sum_{k=1}^{n}e^{i\lambda_k(\bar\alpha_k,\bar\beta_k)t}\frac{2\pi}{\sqrt{-\mathfrak{H}_{\lambda_k}(\bar\alpha_k,\bar\beta_k)}}\int dx\,\Phi_{qp}^{*}(x)\phi_k(\bar\alpha_k,\bar\beta_k,x)\right|^2 , \tag{4.4}$$

where $\mathfrak{H}_{\lambda_k}(\bar\alpha_k,\bar\beta_k)$ is the Hessian of $\lambda_k(\alpha,\beta)$ evaluated at the stationary point $(\bar\alpha_k,\bar\beta_k)$.

From the normalizing condition of the Husimi distribution

$$\int dq\,dp\,\mathcal{H}(q,p,t)=1 , \tag{4.5}$$

it follows that the phase space area significantly occupied by the Husimi distribution grows like t^2. But then $\langle x^2\rangle$ and $\langle p^2\rangle$, and consequently the energy, follow the same behavior.

The application of the stationary phase can be justified for sufficiently well behaved functions ϕ_k. This can be done since the presence of two Bloch phases reduces the problem to a finite dimensional one (and ϕ_k become analytic functions).

Explicitly, suppose $q=4$ $(\alpha=\pi/2)$. Then

$$\mathcal{U}^4=\prod_{j=1}^{4}(e^{ik\cos(\hat x)}e^{-i\hat H_0 T}) , \tag{4.6}$$

where $e^{-i\hat H_0 T}=e^{-i\omega_0 T(\hat a^\dagger\hat a+1/2)}=e^{-i\alpha(\hat n+1/2)}$, which can be written as

$$\mathcal{U}^4=e^{-2i\alpha}e^{-4i\alpha\hat n}\prod_{j=0}^{3}e^{i(4-j)\alpha\hat n}e^{ik\cos(\hat x)}e^{-i(4-j)\alpha\hat n}$$
$$=e^{-2i\alpha}e^{-4i\alpha\hat n}\prod_{j=0}^{3}e^{ik\cos[\hat x\cos(4-j)\alpha+\hat p/\omega_0\sin(4-j)\alpha]} . \tag{4.7}$$

Since $\alpha=\pi/2$ this becomes

$$\mathcal{U}^4=-e^{-2\pi i\hat n}e^{ik\cos(\hat x)}e^{ik\cos(\hat p/\omega_0)}e^{ik\cos(\hat x)}e^{ik\cos(\hat p/\omega_0)}$$
$$=-e^{-2\pi i\hat n}\mathcal{U}_H^2 , \tag{4.8}$$

where $\mathcal{U}_H$ is the well known one period evolution operator for the kicked Harper model (KHM) on the plane with parameters $K=L=k$ and $\hbar=1/\omega_0$. This operator was theoretically investigated in [8], and it was proved that for $\hbar/2\pi=n/m$ $m,n\in\mathbb{Z}$ it can be written as the product of four $m\times m$ matrices. This means that even the case with two Bloch phases can be reduced to a finite dimensional problem. Indeed one has $\omega_0=m/2\pi$, namely, $\hbar=2\pi/m$.

Consider now the case with one Bloch phase. Figures 3 and 4 represent the crystal case $\omega_0/\omega_1=\frac{1}{4}$. Figure 5 is for the triangular crystal case $\omega_0/\omega_1=\frac{1}{3}$. Parameters have been chosen to satisfy, respectively, (3.20a) (Figs. 3 and 4) and (3.22a) (Fig. 5), namely, commutation with a one-parameter group. The observed energy spread is roughly linear in time $\langle E\rangle\sim Dt$ (see Figs. 3 and 5). Even in these cases the Husimi distribution (Fig. 4) spreads roughly isotropically in the phase space. The stationary phase approach would give, in this case, $\mathcal{H}\sim 1/t$ since the decomposition involves just one Bloch phase (and only one integral). Adopting the same line of reasoning one would obtain a phase area $\Delta x\Delta p\sim t$ and, since the propagation remains isotropic, $\langle x^2\rangle$ and $\langle p^2\rangle$ grow like t for large times. Of course, in this case the stationary phase approach is hardly justified since we cannot assume smooth expansion functions.

Our results can be summarized as follows: given a one-dimensional (1D) system invariant along one or two directions in the phase plane, the phase area grows in

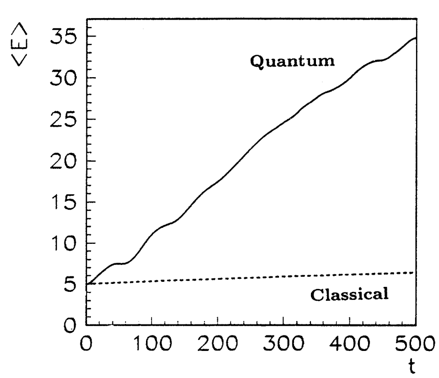

FIG. 3. Average energy versus time for $\omega_0/\omega_1=\frac{1}{4}$, $k/\omega_0=2$, and $\omega_0=0.1$ (diffusive behavior). Full line, quantum; dashed line, classical.

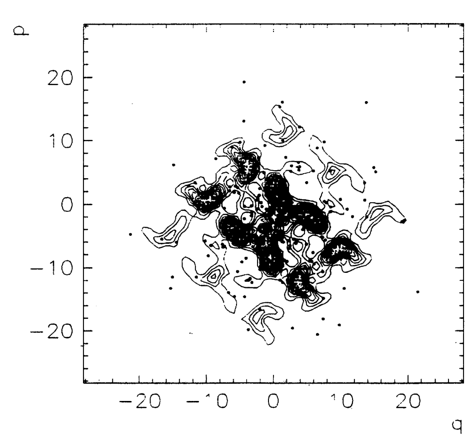

FIG. 4. Comparison between the classical (dots) and the Husimi distribution (contour plot) at the time $t=50$ for $\omega_0/\omega_1=\frac{1}{4}$, $k/\omega_0=2$, and $\omega_0=1$.

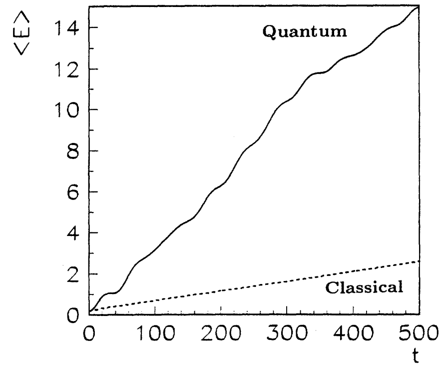

FIG. 5. Average energy versus time for $\omega_0/\omega_1=\frac{1}{3}$, $k/\omega_0=2$, and $\omega_0=0.1$ (diffusive behavior). Full line, quantum; dashed line, classical.

time, respectively, as t or t^2. How are these results connected with the standard ones, regarding, for instance, a particle in a periodic potential? Is it possible to generalize them?

Consider a 3D potential periodic along one direction:

$$V(x,y,z)=V(x+a,y,z) \tag{4.9}$$

without any further specification. Then Bloch's theorem holds and the eigenfunctions are extended within Bloch's bands. This in turn implies $\Delta x(t)\sim t$ with a constant momentum $\Delta p_x(t)\sim\text{const}$.

If the invariance is along two directions x,y (which are independent since they are commuting variables),

$$V(x,y,z)=V(x+a,y+b,z)\ , \tag{4.10}$$

then we have $\Delta x(t)\sim t$, $\Delta y(t)\sim t$, and $\Delta p_x(t)\sim\Delta p_y(t)\sim\text{const}$. In both cases, the projection of the phase space volume along the invariant directions grows in time as t^n, where n is the number of translations. More precisely

$$\Delta x(t)\Delta p_x(t)\sim t \tag{4.11}$$

in the former case, and

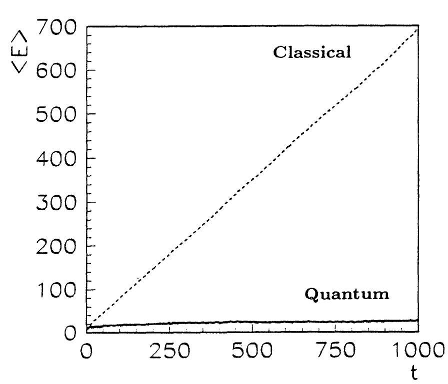

FIG. 6. Average energy versus time for the parameters $\omega_0/\omega_1=\frac{1}{5}$, $k/\omega_0=2$, and $\omega_0=1$ (localization). Full line, quantum; dashed line, classical.

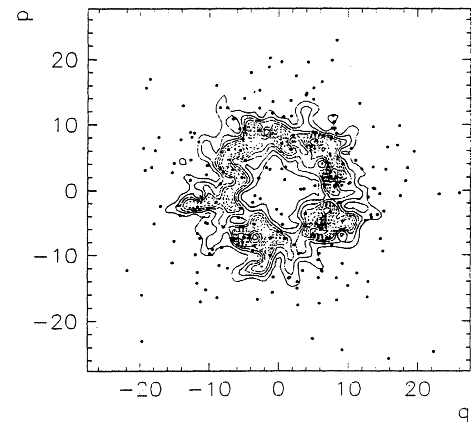

FIG. 7. Comparison between the classical and the Husimi distribution at the time $t=100$ for the same data as in Fig. 6.

$$\Delta x(t)\Delta p_x(t)\Delta y(t)\Delta p_y(t)\sim t^2 \tag{4.12}$$

for the potential (4.10).

It is important to stress that this result does not hold for the coordinate area. Indeed, while it is possible to state that the area in the x-y plane behaves quadratically in time

$$\Delta x(t)\Delta y(t)\sim t^2 \tag{4.13}$$

for the potential (4.10), nothing can be said in the case (4.9). In this last case $\Delta y(t)$ is determined only by the knowledge of the potential.

In any case this is well known. What happens now if the invariance in the usual coordinate space is substituted by the invariance in the phase space?

First, since x and p do not commute, it is necessary to require that the translations be independent [see (3.18)]. Let us consider a 2D system (since its coordinate space is equivalent to the phase space of our 1D system). In the

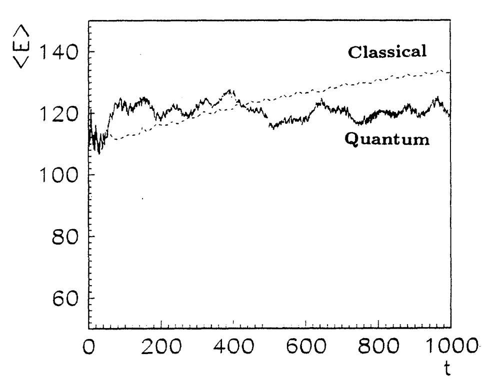

FIG. 8. Average energy versus time for $\omega_0/\omega_1=2/(\sqrt{5}+1)$, $k/\omega_0=1$, and $\omega_0=1$ (localization). Full line, quantum; dashed line, classical. The initial classical and quantum distributions are peaked around the point (0,15) in the classical phase plane. For smaller energy, the web is not connected and the classical diffusion is forbidden.

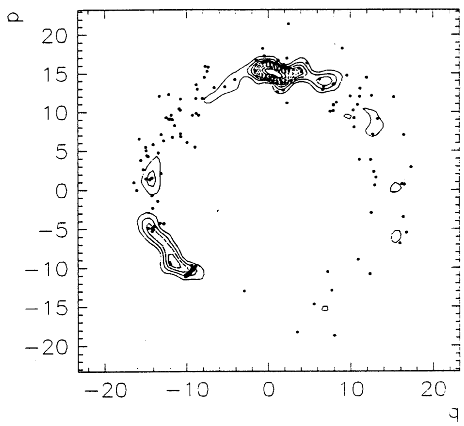

FIG. 9. Comparison between the classical and the Husimi distribution at the time $t=300$ for the same data as in Fig. 8.

presence of two Bloch phases, the standard results (4.13) would predict an area growing as t^2, as we have $(\Delta x(t)\Delta p_x(t)\sim t^2)$. In the other case (one translation) no prediction can be done about $\Delta x(t)\Delta y(t)$ while we found $\Delta x(t)\Delta p_x(t)\sim t$. This apparent disagreement [justified on the basis of (4.11)] is related to the fact that x and p_x are not independent variables.

It is also important to remark the different propagation of the phase area for the kicked harmonic oscillator compared with the 1D particle in a periodic potential (or 1D free particle). Despite the fact that $\Delta x(t)\Delta p_x(t)\sim t$ in both cases, we have $\Delta x(t)\sim\Delta p_x(t)\sim t^{1/2}$ in the former model (due to the isotropic propagation) and $\Delta x(t)\sim t$, $\Delta p_x(t)\sim$const in the latter one. One can imagine a homogeneous propagation in a square or inside a tube (with the same phase area growth), respectively.

This important result has been obtained for the QKHO. As explained before [see Eq. (4.8)] this model can be put in close correspondence (when $q=4$ and $K=L=k$) with the KHM on the plane, which has been considered in [8]. A general relation between the QKHO and the KHM has also been considered quite recently in [9]. For this model diffusive and ballistic behaviors have been found when $\hbar/2\pi$ is irrational or rational, respectively [8]. However, a lot of work has been reported in the literature for the KHM on the cylinder (see, for instance, [10]). Among these results we mention the anomalous diffusion (with an exponent different from 1) observed in [11] for $K=L$ and irrational $\hbar/2\pi$. We are then left with two possibilities:

(i) The models are different and there is no way to connect spreading on the cylinder with that on the plane.

(ii) Suppose one is able to prove that the energy spreading has the same power law for both models. This would imply that the stationary phase argument cannot be applied for the one-parameter group.

In any case these points deserve future investigation.

Is it possible to extrapolate a general theorem from the previous results? Probably only in the weak formulation: if the stationary phase argument can be applied, then the results follow. In some sense the applicability of the stationary phase method is crucial. The opposite is certainly not true, in the sense that diffusion and anomalous diffusion can be found even in models without translational invariance, and it is known that these kinds of propagations are related to the spectral properties of the Hamiltonian [12].

We also made a few additional calculations for frequencies ratios $\omega_0/\omega_1=1/q$, with $q\notin q_c$, and ω_0/ω_1 irrational. These cases were not taken into account in our theoretical scheme. In both cases we observe quantum localization despite the classical diffusion. These results are shown in Figs. 6 and 7 for $\omega_0/\omega_1=\frac{1}{5}$ and in Figs. 8 and 9 for $\omega_0/\omega_1=2/(\sqrt{5}+1)$. However, evidence was found [3] for a transition from localized to delocalized states for $\omega_0/\omega_1=\frac{1}{5}$. We may therefore assert that this effect is not connected with the translational invariance of the quasienergy operator.

V. CONCLUSIONS

We have studied the quantum version of the kicked harmonic oscillator in the crystal case, where its classical version is known to exhibit diffusive motion [1]. We have found that the corresponding quantum system behaves diffusively or ballistically in the phase plane. This is quite surprising in the study of kicked systems; see, e.g., the prototype: the kicked rotator [13], where the quantum analogue of classical diffusion was localization or ballistic motion. While ballistic motion, as in the kicked rotator example, can be obtained by a peculiar combination of the parameters (quantum resonance [14]), quantum diffusion is a feature of this model. These behaviors can both be traced back to the translational properties of the system itself by using numerical suggestions as well as analytical estimates.

Future investigations will regard the quasicrystal quantum model: in such a case the translational invariance is destroyed but a new symmetry related to the spectral properties of the Floquet operator appears.

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